MRC Technical Summary Report #2103

A THEOREM CONCERNING UNIFORM
SIMPLIFICATION AT A TRANSITION POINT
AND THE PROBLEM OF RESONANCE

Yasutaka Sibuya

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

August 1980

(Received June 13, 1980)

Approved for public release
Distribution unlimited

80 12 22 067

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

A THEOREM CONCERNING UNIFORM SIMPLIFICATION AT A TRANSITION

POINT AND THE PROBLEM OF RESONANCE[*]

Yasutaka Sibuya[±]

Technical Summary Report #2103
August 1980

ABSTRACT

Given sectors $S_j = \{\varepsilon; a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho\}$ $(1 \leq j \leq \nu)$ and

functions δ_j $(1 \leq j \leq \nu)$ such that (i) $\bigcup_j S_j = \{\varepsilon; 0 < |\varepsilon| < \rho\}$, (ii) δ_j

is holomorphic in S_j, (iii) δ_j is asymptotically zero as $\varepsilon \to 0$ in S_j,

(iv) $|\delta_j(\varepsilon) - \delta_k(\varepsilon)| \leq c_0 \exp\{-c_1/|\varepsilon|^\lambda\}$ in $S_j \cap S_k$ for some positive

numbers c_0, c_1 and λ whenever $S_j \cap S_k \neq \emptyset$, we prove that

$|\delta_j(\varepsilon)| \leq c_2 \exp\{-c_1/|\varepsilon|^\lambda\}$ in S_j for some positive number c_2. Then,

utilizing this result, we prove that Matkowsky-condition implies the resonance

in the sense of N. Kopell under a reasonable assumption. The sufficiency of

Matkowsky-condition with regard to the Ackerberg-O'Malley resonance has been

an open question. This work gives an affirmative answer to this question in

a reasonably general case.

AMS (MOS) Subject Classifications: 30B40, 30E15, 33A40, 34E20

Key Words: Analytic continuation, Asymptotic representations in the complex
            domain, Parabolic cylinder functions, Singular perturbations,
            Turning point theory

Work Unit Number 1 (Applied Analysis)

SIGNIFICANCE AND EXPLANATION

The basic question is as follows:

Consider a second-order linear differential equation:

(1) $\epsilon x'' + F(x, \ldots)v' + \ldots = 0$

under some reasonable assumptions on F and G. Let v(x,ε) be a solution.

Then, generally speaking, $\lim_{\epsilon \to 0} v(x,\epsilon)$ satisfies the first order equation:

(2)   F(x,0)dv/dx + G(x,0)v = 0.

The problem of finding the relation between (1) and (2) is called the problem

of singular perturbations. "singular" means that solutions of the equation (2)

do not contain as many free parameters as solutions of (1) do. In other words,

we lose something. This phenomenon is explained by means of

boundary-layers. Naturally, various physical phenomena exhibit similar

behaviours. Therefore, the problem of singular perturbations has been studied

for many years.

In a certain situation which arises naturally in applications,

$\lim v$ as ϵ then approaches 0 $(x, \epsilon) = 0$

$\lim v(x,\epsilon) = 0$ identically for practically all the solutions v of (1),

except when F and G are related in a specific way. This exceptional case

is called the case of resonance. It is important to find an effectively

computable condition for the resonance. B. J. Matkowsky found such a condition.

However, so far, it has been mathematically very difficult to prove that the

Matkowsky-condition actually guarantees the resonance. The difficulty is due

to the fact that a quantity which is decisive in determining the resonance is

so small that any existing mathematical tool has failed to dig this quantity

out of the differential equation clearly. In this work, we shall provide

such a tool.

B

A THEOREM CONCERNING UNIFORM SIMPLIFICATION AT A TRANSITION
POINT AND THE PROBLEM OF RESONANCE[*]

Yasutaka Sibuya[†]

1. <u>Introduction</u>: The main result of this paper is the following theorem:

<u>Theorem 1.1</u>: <u>Let</u>

(1.1) $\qquad S_j = \{\varepsilon; \ a_j < \arg \varepsilon < b_j, \ 0 < |\varepsilon| < \rho\}$ $\qquad (j = 1,\ldots,\nu)$

<u>be sectors in the complex ε-plane, where</u> ρ <u>is a positive number and the</u> a's

<u>and the</u> b's <u>are real numbers.</u> <u>Let</u> $\delta_1(\varepsilon),\ldots,\delta_\nu(\varepsilon)$ <u>be functions of</u> ε.

<u>Assume that</u>

(i) $\quad S_1 \cup S_2 \cup \cdots \cup S_\nu = \{\varepsilon; \ 0 < |\varepsilon| < \rho\}$;

(ii) $\quad \delta_j(\varepsilon)$ <u>is holomorphic in</u> S_j;

(iii) $\quad \delta_j(\varepsilon)$ <u>is asymptotically zero as</u> $\varepsilon \to 0$ <u>in</u> S_j, i.e.

$\qquad |\delta_j(\varepsilon)| \leq K_N |\varepsilon|^N$ $\qquad (N = 0,1,\ldots)$ <u>in</u> S_j

<u>for some positive numbers</u> K_N;

(iv) $\quad$ <u>if</u> $S_j \cap S_k \neq \phi$, <u>we have</u>

(1.2) $\qquad |\delta_j(\varepsilon) - \delta_k(\varepsilon)| \leq c_0 \exp(-c_1/|\varepsilon|^\lambda)$ <u>in</u> $S_j \cap S_k$,

<u>for some positive numbers</u> $c_0, \ c_1$ <u>and</u> λ.

<u>Then, there exists a positive number</u> H <u>such that</u>

(1.3) $\qquad |\delta_j(\varepsilon)| \leq H \exp(-c_1/|\varepsilon|^\lambda)$ <u>in</u> S_j, $\qquad j = 1,2,\ldots,\nu$.

We shall prove this theorem in Section 8. (For another proof, see J.-P. Ramis [5; Theorem 11-(i), p. 189].) In other sections, utilizing Theorem 1.1, we shall treat the following problem:

We consider a differential equation:

$$(1.4) \qquad \varepsilon d^2 v/dx^2 + F(x,\varepsilon)dv/dx + G(x,\varepsilon)v = 0 ,$$

where F and G are holomorphic in two complex variables x and ε in a domain:

$$(1.5) \qquad x \in \mathcal{D}_0, \quad |\varepsilon| < \rho_0 ,$$

where $\mathcal{D}_0$ is a domain in the x-plane and ρ_0 is a positive number. We assume that $\mathcal{D}_0$ contains a real interval:

$$(1.6) \qquad I_0 = \{x; -a \leq \operatorname{Re}(x) \leq b, \operatorname{Im}(x) = 0\} ,$$

where a and b are positive numbers. We also assume that

$$(1.7) \qquad F(x,0) = -2x .$$

We say that the differential equation (1.4) satisfies Matkowsky-condition, if there exists a non-trivial formal power series solution of (1.4):

$$(1.8) \qquad v = \sum_{m=0}^{\infty} a_m(x)\varepsilon^m$$

such that all the $a_m(x)$ are bounded on the real interval I_0. We also say that the differential equation (1.4) exhibits a resonance in the sense of N. Kopell on I_0 if there exists a solution $v(x,\varepsilon)$ satisfying $v(b,\varepsilon) = 1$, such that $v(x,\varepsilon)$ converges uniformly on I_0 as $\varepsilon \to +0$ to a non-trivial solution of

$$(1.9) \qquad F(x,0)dv/dx + G(x,0)v = 0 .$$

(Cf. B. J. Matkowsky [4] and N. Kopell [2].)

We shall prove the following theorem:

THEOREM 1.2. If $\mathcal{D}_0$ is a disk with the center at $x = 0$, i.e.

(1.10) $$\mathcal{D}_0 = \{x;\ |x| < r_0\} \text{ for some } r_0 > 0.$$

Then, Matkowsky-condition implies the resonance in the sense of N. Kopell.

In our argument, the assumption that F and G are holomorphic in (x, ε) in a poly-disk (1.5) is indispensable. In our proof, we follow roughly the guide-line given by R. McKelvey and R. Bohac [3]. It seems to us that our results yield a sharp estimate for eigenvalues studied by P. P. N. de Groen [1]. In Section 2, we discuss a more general case.

Throughout this research, the author enjoyed lively discussions with N. Kopell, B. J. Matkowsky and P. P. N. de Groen.

2. <u>A standard form</u>: Let ρ_0 be a positive number and let $\mathcal{D}$ be a domain in the complex ξ-plane which contains a real interval

$$(2.1) \qquad I = \{\xi;\ -\alpha \leq \text{Re}(\xi) \leq \beta,\ \text{Im}(\xi) = 0\}\ ,$$

where α and β are positive numbers.

We shall consider a linear differential equation:

$$(2.2) \qquad \varepsilon d^2 v/d\xi^2 + f(\xi,\varepsilon)dv/d\xi + g(\xi,\varepsilon)v = 0\ ,$$

where f and g are holomorphic in two variables ξ and ε in the domain

$$(2.3) \qquad \xi \in \mathcal{D},\ \ |\varepsilon| < \rho_0\ .$$

Set

$$(2.4) \qquad f_0(\xi) = f(\xi,0)\ .$$

We assume that

$$(2.5) \qquad f_0(0) = 0,\ \ \ f_0'(0) \neq 0\ ,$$

$$(2.6) \qquad \xi f_0(\xi) < 0\ \ \text{for}\ \xi \in I,\ \ \text{if}\ \xi \neq 0\ .$$

Under this situation, we can write f_0 as

$$(2.7) \qquad f_0(\xi) = \xi h(\xi)\ ,$$

where $h(\xi)$ is holomorphic in $\mathcal{D}$ and

$$(2.8) \qquad h(\xi) < 0\ \ \text{for}\ \xi \in I\ .$$

Let us change the independent variable by

$$(2.9) \qquad x = \varphi(\xi) = \{-\int_0^\xi f_0(t)dt\}^{\frac{1}{2}}\ .$$

Then, (2.2) becomes

$$(2.10) \qquad \varepsilon d^2 v/dx^2 + F(x,\varepsilon)dv/dx + G(x,\varepsilon)v = 0\ ,$$

where

$$(2.11) \qquad F(\varphi,\varepsilon) = (\varphi')^{-2}\{\varphi'f + \varepsilon\varphi''\},\ \ G(\varphi,\varepsilon) = (\varphi')^{-2}g\ .$$

Since $f_0 = -2\varphi\varphi'$, we have

$$(2.12) \qquad F(x,\varepsilon) = -2x + \varepsilon k(x,\varepsilon)\ ,$$

-4-

and $k(x,\varepsilon)$ and $G(x,\varepsilon)$ are holomorphic in a domain

(2.13)
$$x \in \mathcal{D}_0, \qquad |\varepsilon| < \rho_0 ,$$

where $\mathcal{D}_0$ is a domain in the x-plane which contains the real interval:

(2.14)
$$I_0 = \{x;\ -a \leq \mathrm{Re}(x) \leq b,\ \mathrm{Im}(x) = 0\} ,$$

where

(2.15)
$$a = \sqrt{-\int_0^{-\alpha} f_0(t)\,dt}, \qquad b = \sqrt{-\int_0^{\beta} f_0(t)\,dt} .$$

Another transformation:

(2.16)
$$v = w \exp\{-\frac{1}{2\varepsilon} \int_0^x F(t,\varepsilon)\,dt\}$$

takes (2.10) to

(2.17)
$$\varepsilon^2 d^2w/dx^2 - \{\frac{1}{4} F(x,\varepsilon)^2 + \varepsilon(\frac{1}{2} \frac{\partial F}{\partial x}(x,\varepsilon) - G(x,\varepsilon))\}w = 0 .$$

Note that

(2.18)
$$\frac{1}{4} F^2 + \varepsilon(\frac{1}{2} \partial F/\partial x - G) = x^2 + \varepsilon R(x,\varepsilon) ,$$

where R is holomorphic in (2.13).

<u>Remark</u>: To find the domain $\mathcal{D}_0$, we must take into account not only singularities of f and g, but also singularities of φ, i.e. the transformation (2.9). In particular, any zeros of f_0 would yield branch-points with respect to x.

3. **Formal simplification**: It is known that there exist three formal power
series in ε:

$$(3.1) \qquad A(x,\varepsilon) = \sum_{m=0}^{\infty} A_m(x)\varepsilon^m \; ,$$

$$(3.2) \qquad B(x,\varepsilon) = \sum_{m=0}^{\infty} B_m(x)\varepsilon^m \; ,$$

and

$$(3.3) \qquad C(\varepsilon) = \sum_{m=0}^{\infty} C_m\varepsilon^m$$

such that

(i) $A_m(x)$ and $B_m(x)$ are holomorphic in the domain $\mathcal{D}_0$;

(ii) C_m are constants;

(iii) the formal transformation:

$$(3.4) \qquad w = A(x,\varepsilon)u + B(x,\varepsilon)(\varepsilon du/dx)$$

takes (2.17) to

$$(3.5) \qquad \varepsilon^2 d^2u/dx^2 - \{x^2 + \varepsilon C(\varepsilon)\}u = 0 \; ;$$

(iv) we have

$$(3.6) \qquad A_0(x)^2 - (xB_0(x))^2 = 1 \quad \text{identically in } \mathcal{D}_0 \; .$$

To effect the transformation (3.4), we differentiate both sides of (3.4)
with respect to x. Then, we derive

$$(3.7) \qquad \varepsilon dw/dx = (\varepsilon A' + (x^2 + \varepsilon C)B)u + (A + \varepsilon B')(\varepsilon du/dx) \; ,$$

and

$$(3.8) \qquad \varepsilon^2 d^2w/dx^2 = (\varepsilon(\varepsilon A' + (x^2 + \varepsilon C)B)' + (x^2 + \varepsilon C)(A + \varepsilon B'))u$$
$$+ ((\varepsilon A' + (x^2 + \varepsilon C)B) + \varepsilon(A + \varepsilon B')')(\varepsilon du/dx) \; ,$$

where $'$ denotes $\partial/\partial x$. Since $\varepsilon^2 d^2w/dx^2 = (x^2 + \varepsilon R)w$, we derive the
following equations on A, B and C:

$$(3.9) \quad \begin{cases} (x^2 + \varepsilon R)A = \varepsilon(\varepsilon A' + (x^2 + \varepsilon C)B)' + (x^2 + \varepsilon C)(A + \varepsilon B') , \\[2mm] (x^2 + \varepsilon R)B = (\varepsilon A' + (x^2 + \varepsilon C)B) + \varepsilon(A + \varepsilon B')' . \end{cases}$$

In particular, if we put

$$X = A_0, \qquad Y = xB_0 ,$$

we have

$$dX/dx = \frac{R_0(x) - C_0}{2x} Y, \qquad dY/dx = \frac{R_0(x) - C_0}{2x} X ,$$

where $R_0(x) = R(x,0)$. Hence

$$d(X^2 - Y^2)/dx = 0 \quad \text{identically.}$$

Choose $C_0 = R_0(0)$ and the initial condition: $X(0) = 1$, $Y(0) = 0$. Then, we can determine A_0, B_0 and C_0 so that (3.6) is satisfied. Other coefficients A_m, B_m and C_m can be determined in a similar way.

By virtue of (3.6), we can solve (3.4) and (3.7) with respect to u and $\varepsilon du/dx$:

$$(3.10) \quad \begin{cases} u = E_{11}(x,\varepsilon)w + E_{12}(x,\varepsilon)(\varepsilon dw/dx) , \\[2mm] \varepsilon du/dx = E_{21}(x,\varepsilon)w + E_{22}(x,\varepsilon)(\varepsilon dw/dx) , \end{cases}$$

where E_{jk} are formal power series in ε whose coefficients are holomorphic in $\mathcal{D}_0$. In particular,

$$(3.11) \quad \begin{cases} E_{11}(x,0) = E_{22}(x,0) = A_0(x) , \\[2mm] E_{12}(x,0) = -B_0(x), \quad E_{21}(x,0) = -x^2 B_0(x) . \end{cases}$$

Note that

$$(3.12) \qquad C_0 = R_0(0) = -1 + 2\,\frac{g(0,0)}{f_0'(0)} .$$

-7-

4. Outer expansions: A formal power series in ϵ:

(4.1)
$$v = \sum_{m=0}^{\infty} a_m(x)\,\epsilon^m$$

is called an outer expansion associated with the differential equation (2.10),
if (4.1) formally satisfies (2.10). The power series (4.1) is an outer
expansion if and only if

(4.2)
$$\begin{cases} -2x\,da_0/dx + G_0(x)a_0 = 0 \ , \\ -2x\,da_m/dx + G_0(x)a_m = L_m(x) - d^2a_{m-1}(x)/dx^2 \qquad (m \geq 1) \ , \end{cases}$$

where $G_0(x) = G(x,0)$ and $L_m(x)$ is linear homogeneous in $a_0,\ldots,a_{m-1}$ and
$da_0/dx,\ldots,da_{m-1}/dx$ with coefficients holomorphic in $\mathcal{D}_0$.

DEFINITION 4.1: The differential equation (2.10) is said to satisfy Matkowsky-
condition, if there exists a non-trivial outer expansion (4.1) such that all
the $a_m(x)$ are bounded on the real interval I_0 (cf. (2.14)).

LEMMA 4.2: The differential equation (2.10) satisfies Matkowsky-condition if
and only if C_0 is a negative odd integer and

(4.3)
$$C_m = 0 \qquad (m \geq 1) \ .$$

Proof: The transformation

(4.4)
$$u = y\,\exp\{-x^2/(2\epsilon)\}$$

changes (3.5) to

(4.5)
$$\epsilon d^2y/dx^2 - 2x\,dy/dx - (1 + C)y = 0 \ .$$

By a straight-forward computation, we can prove that the differential equa-
tion (4.5) satisfies Matkowsky-condition if and only if C_0 is a negative
odd integer and $C_m = 0$ for $m \geq 1$.

Note also that, if all the a_m are bounded, then all the da_m/dx are
bounded. Otherwise, d^2a_m/dx^2 would have much worse singularities at $x = 0$,
and hence a_{m+1} would be unbounded (cf. (4.2)).

Finally, by manipulating with the transformations (2.16), (3.4) and (3.7), and (3.10) together with (4.4), we can show that the differential equation (2.10) satisfies Matkowsky-condition if and only if the differential equation (4.5) satisfies the same condition. This completes the proof of Lemma 4.2.

5. Uniform simplification: Hereafter, we shall assume that

(5.1) $\qquad c_0 = -p$, where p is a positive odd integer ;

(5.2) $\qquad\qquad c_m = 0$ for $m \geq 1$:

(5.3) $\qquad\qquad \mathcal{D}_0 = \{x; \ |x| < r_0\}$ for some $r_0 > 0$.

The assumption (5.3) means that $\mathcal{D}_0$ is a disk of radius r_0 with the center

at $x = 0$.

Let us choose two positive numbers r_1 and r such that

(5.4) $\qquad\qquad\qquad 0 < r_1 < r < r_0$

and that the disk

(5.5) $\qquad\qquad\qquad \mathcal{D}_1 = \{x; \ |x| < r_1\}$

contains the real interval I_0 (cf. (2.14)).

Let us denote by $T(x,\varepsilon)$ the two-by-two matrix:

(5.6)
$$\begin{bmatrix} A(x,\varepsilon) & B(x,\varepsilon) \\ \varepsilon A'(x,\varepsilon) + (x^2 - \varepsilon p)B(x,\varepsilon) & A(x,\varepsilon) + \varepsilon B'(x,\varepsilon) \end{bmatrix}$$

(cf. (3.4) and (3.7)). Set

(5.7)
$$U = \begin{bmatrix} u \\ \varepsilon du/dx \end{bmatrix}, \qquad W = \begin{bmatrix} w \\ \varepsilon dw/dx \end{bmatrix} .$$

Then, the formal transformation

(5.8) $\qquad\qquad\qquad W = T(x,\varepsilon)U$

takes the system

(5.9)
$$\varepsilon dW/dx = \begin{bmatrix} 0 & 1 \\ x^2 + \varepsilon R(x,\varepsilon) & 0 \end{bmatrix}W$$

to

(5.10)
$$\varepsilon dU/dx = \begin{bmatrix} 0 & 1 \\ x^2 - \varepsilon p & 0 \end{bmatrix}U .$$

The inverse of the matrix $T(x,\varepsilon)$ is given by

$$(5.11) \qquad T(x,\varepsilon)^{-1} = \begin{bmatrix} E_{11}(x,\varepsilon) & E_{12}(x,\varepsilon) \\ E_{21}(x,\varepsilon) & E_{22}(x,\varepsilon) \end{bmatrix}$$

(cf. (3.10)).

Set

$$(5.12) \qquad \mathcal{D}_2 = \{x; \ |x| < r\} \ .$$

It is known that there exist two positive numbers ρ_1 and ρ_2, a function $\delta(\varepsilon)$, and a two-by-two matrix $P(x,\varepsilon)$ such that

(i) $\delta(\varepsilon)$ is holomorphic in the sector

$$(5.13) \qquad S = \{\varepsilon; \ |\arg \varepsilon| < \rho_1, \ 0 < |\varepsilon| < \rho_2\} \ ;$$

(ii) $\delta(\varepsilon)$ is asymptotically zero as $\varepsilon \to 0$ in S, i.e.

$$(5.14) \qquad |\delta(\varepsilon)| \leqq K_N |\varepsilon|^N \qquad (N = 0,1,2,\dots) \quad \text{in} \ S$$

for some positive numbers K_N;

(iii) entries of P and P^{-1} are holomorphic in the domain

$$(5.15) \qquad x \in \mathcal{D}_2, \qquad \varepsilon \in S \ ;$$

(iv) P (resp. P^{-1}) admits the matrix T (resp. T^{-1}) as an asymptotic expansion as $\varepsilon \to 0$ in S which is valid uniformly in $\mathcal{D}_2$;

(v) the transformation

$$(5.16) \qquad W = P(x,\varepsilon)V$$

takes (5.9) to

$$(5.17) \qquad \varepsilon dV/dx = \begin{bmatrix} 0 & 1 \\ x^2 - \varepsilon(p + \delta(\varepsilon)) & 0 \end{bmatrix} V$$

in the domain (5.15). (Cf. Y. Sibuya [6].)

Utilizing this result and manipulating with rotations of the disk $\mathcal{D}_2$, we can prove the following lemma:

LEMMA 5.1: There exist sectors

$$(5.18-j) \qquad S_j = \{\epsilon;\ a_j < \arg \epsilon < b_j,\ 0 < |\epsilon| < \rho_3\} \qquad (j = 1,2,\ldots,k)\ ,$$

(where ρ_3 is a positive number and the a's and the b's are real

numbers), functions $\delta_1(\epsilon),\ldots,\delta_k(\epsilon)$, and two-by-two matrices $P_1(x,\epsilon),\ldots,$

$P_k(x,\epsilon)$ such that $S_1 \cup \cdots \cup S_k = \{\epsilon;\ 0 < |\epsilon| < \rho_3\}$ and that

  (i) $\quad \delta_j(\epsilon)$ is holomorphic in S_j;

  (ii) $\quad \delta_j(\epsilon)$ is asymptotically zero as $\epsilon \to 0$ in S_j ;

(iii) $\quad$ entries of P_j and P_j^{-1} are holomorphic in the domain

$$(5.19-j) \qquad\qquad\qquad x \in \mathcal{D}_2,\qquad \epsilon \in S_j\ ;$$

  (iv) $\quad P_j$ (resp. P_j^{-1}) admits the matrix T (resp. T^{-1}) as an

asymptotic expansion as $\epsilon \to 0$ in S_j which is valid uniformly in $\mathcal{D}_2$;

  (v) $\quad$ the transformation

$$(5.20) \qquad\qquad\qquad W = P_j(x,\epsilon)V_j$$

takes (5.9) to

$$(5.21-j) \qquad\qquad \epsilon dV_j/dx = \begin{bmatrix} 0 & 1 \\ x^2 - \epsilon(p + \delta_j(\epsilon)) & 0 \end{bmatrix} V_j$$

in the domain (5.19-j).

6. **An estimate for** $\delta_j(\varepsilon)$. In this section, as an application of our main theorem (cf. Theorem 1.1), we shall derive an estimate

$$(6.1) \qquad |\delta_j(\varepsilon)| \leq H_j \exp(-r^2/|\varepsilon|) \quad \text{for} \quad \varepsilon \in S_j ,$$

where H_j is a positive number. To do this, it is sufficient to prove that, if $S_\ell \cap S_j \neq \emptyset$, we have

$$(6.2) \qquad |\delta_\ell(\varepsilon) - \delta_j(\varepsilon)| \leq M_{\ell j} \exp(-r^2/|\varepsilon|) \quad \text{for} \quad \varepsilon \in S_\ell \cap S_j ,$$

where $M_{\ell j}$ is a positive number. To derive an estimate (6.2), we need some preparation.

Let us consider the differential equation

$$(6.3) \qquad d^2 z/dt^2 - (t^2 - a)z = 0, \quad \text{where} \quad a \quad \text{is a parameter} .$$

This equation admits a solution

$$(6.4) \qquad\qquad\qquad z = Z(t,a)$$

such that

   (i) Z is an entire function in (t,a) ;

   (ii) $\displaystyle\lim_{t\to+\infty} t^{\frac{1}{2}(1-a)} e^{\frac{1}{2}t^2} Z(t,a) = 1$

uniformly in a if a is in a compact set in the a-plane.

The solution $Z(t,a)$ is uniquely determined by (i) and (ii). The functions $Z((-i)t,-a)$, $Z(-t,a)$, and $Z(it,-a)$ are also solutions of (6.3). Set

$$(6.5\text{-}0) \qquad \Psi_0(t,a) = \begin{bmatrix} Z(t,a) & Z((-i)t,-a) \\ Z'(t,a) & (-i)Z'((-i)t,-a) \end{bmatrix} ,$$

$$(6.5\text{-}1) \qquad \Psi_1(t,a) = \begin{bmatrix} Z((-i)t,-a) & Z(-t,a) \\ (-i)Z'((-i)t,-a) & -Z'(-t,a) \end{bmatrix} ,$$

$$(6.5\text{-}2) \qquad \Psi_2(t,a) = \begin{bmatrix} Z(-t,a) & Z(it,-a) \\ -Z'(-t,a) & iZ'(it,-a) \end{bmatrix} ,$$

and

$$(6.5-(-1)) \qquad \Psi_{-1}(t,a) = \begin{bmatrix} Z(it,-a) & Z(t,a) \\ iZ'(it,-a) & Z'(t,a) \end{bmatrix} ,$$

where $'$ denotes $\partial/\partial t$. These four matrices are matrices of independent solutions of (6.3).

Set

$$(6.6) \qquad \lambda_1(a) = 2^{-\frac{1}{2}a} e^{\frac{1}{4}\pi i(a+1)} \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}(1-a))} , \quad \lambda_2(a) = (-i)e^{\frac{1}{2}a\pi i} ,$$

and

$$(6.7) \qquad C(a) = \begin{bmatrix} \lambda_1(a) & 1 \\ \lambda_2(a) & 0 \end{bmatrix} .$$

Then,

$$(6.8) \qquad \begin{cases} \Psi_0(t,a) = \Psi_1(t,a)C(a), & \Psi_1(t,a) = \Psi_2(t,a)C(-a) , \\ \Psi_2(t,a) = \Psi_{-1}(t,a)C(a), & \Psi_{-1}(t,a) = \Psi_0(t,a)C(-a) . \end{cases}$$

Fix i and j so that $S_i \cap S_j \neq \emptyset$. Choose a branch of $\cdot^{\frac{1}{2}}$ in the sector $S_i \cap S_j$. Set

$$(6.9) \qquad \Delta(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{\frac{1}{2}} \end{bmatrix} ,$$

and

$$(6.10-h) \qquad \begin{cases} \Phi_{i,h}(x,\varepsilon) = \Delta(\varepsilon)\Psi_h(x/\varepsilon^{\frac{1}{2}}, p + \delta_i(\varepsilon)), \\ \Phi_{j,h}(x,\varepsilon) = \Delta(\varepsilon)\Psi_h(x/\varepsilon^{\frac{1}{2}}, p + \delta_j(\varepsilon)), \end{cases} \qquad (h = -1,0,1,2) .$$

Then, $\Phi_{i,0}(x,\varepsilon)$, $\Phi_{i,1}(x,\varepsilon)$, $\Phi_{i,2}(x,\varepsilon)$ and $\Phi_{i,-1}(x,\varepsilon)$ (resp. $\Phi_{j,0}(x,\varepsilon)$, $\Phi_{j,1}(x,\varepsilon)$, $\Phi_{j,2}(x,\varepsilon)$ and $\Phi_{j,-1}(x,\varepsilon)$) are fundamental matrix solutions of (5.21-i) (resp. (5.21-j)) such that

$$(6.11\text{-}\ell) \quad \begin{cases} \phi_{\ell,0}(x,\epsilon) = \phi_{\ell,1}(x,\epsilon)\mathcal{C}(p + \delta_\ell(\epsilon)) \,, \\[2mm] \phi_{\ell,1}(x,\epsilon) = \phi_{\ell,2}(x,\epsilon)\mathcal{C}(-p - \delta_\ell(\epsilon)) \,, \\[2mm] \phi_{\ell,2}(x,\epsilon) = \phi_{\ell,-1}(x,\epsilon)\mathcal{C}(p + \delta_\ell(\epsilon)) \,, \\[2mm] \phi_{\ell,-1}(x,\epsilon) = \phi_{\ell,0}(x,\epsilon)\mathcal{C}(-p - \delta_\ell(\epsilon)) \,, \end{cases}$$

and

$$(6.11\text{-}j) \quad \begin{cases} \phi_{j,0}(x,\epsilon) = \phi_{j,1}(x,\epsilon)\mathcal{C}(p + \delta_j(\epsilon)) \,, \\[2mm] \phi_{j,1}(x,\epsilon) = \phi_{j,2}(x,\epsilon)\mathcal{C}(-p - \delta_j(\epsilon)) \,, \\[2mm] \phi_{j,2}(x,\epsilon) = \phi_{j,-1}(x,\epsilon)\mathcal{C}(p + \delta_j(\epsilon)) \,, \\[2mm] \phi_{j,-1}(x,\epsilon) = \phi_{j,0}(x,\epsilon)\mathcal{C}(-p - \delta_j(\epsilon)) \,. \end{cases}$$

Set

$$(6.12) \qquad\qquad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \,,$$

and

$$(6.13) \quad \begin{cases} \mathcal{Q}_{\ell,h}(x,\epsilon) = \phi_{\ell,h}(x,\epsilon)\exp\{(-1)^h \dfrac{x^2}{2\epsilon} J\} \,, \\[3mm] \mathcal{Q}_{j,h}(x,\epsilon) = \phi_{j,h}(x,\epsilon)\exp\{(-1)^h \dfrac{x^2}{2\epsilon} J\} \,, \end{cases} \qquad (h = -1,0,1,2) \,.$$

It is known that, if (x,ϵ) is in a domain

$$(6.14\text{-}h) \qquad x \in \mathcal{D}_2, \; \epsilon \in S_\ell \cap S_j, \; \left| \arg\left(\frac{x}{\epsilon}\right) - \frac{1}{4}\pi - \frac{1}{2}h\pi \right| \le \frac{1}{2}\pi - \nu \,,$$

where ν is a small positive number, we have

$$(6.15) \qquad \|\mathcal{Q}_{j,h}(x,\epsilon)\| \le H|\epsilon|^q, \qquad \|\mathcal{Q}_{j,h}(x,\epsilon)^{-1}\| \le H|\epsilon|^q \,,$$

where H is a positive number depending on ν, q is a real number, and $\|\ \|$ denotes a usual norm of matrices. Furthermore, the matrix

$$(6.16) \qquad\qquad \mathcal{Q}_{j,h}(x,\epsilon) - \mathcal{Q}_{\ell,h}(x,\epsilon)$$

is asymptotically zero as $\epsilon \to 0$ in $S_\ell \cap S_j$ uniformly in the domain

$(6.14\text{-}h)$. (For those results, see, for example, Y. Sibuya [6,7].)

Let $P_\ell(x,\epsilon)$ and $P_j(x,\epsilon)$ be the matrices given in Lemma 5.1. Then, $P_\ell(x,\epsilon)\Phi_{\ell,0}(x,\epsilon)$ and $P_j(x,\epsilon)\Phi_{j,0}(x,\epsilon)$ are two fundamental matrix solutions of (5.9) in the domain

(6.17)
$$x \in \mathcal{D}_2, \qquad \epsilon \in S_\ell \cap S_j .$$

Therefore, there exists a two-by-two matrix $L(\epsilon)$ such that

(6.18)
$$P_\ell(x,\epsilon)\Phi_{\ell,0}(x,\epsilon) = P_j(x,\epsilon)\Phi_{j,0}(x,\epsilon)L(\epsilon) .$$

Note that $L(\epsilon)$ does not depend on x. It follows from (6.18) that

(6.19)
$$\exp\{-\frac{x^2}{2\epsilon} J\}L(\epsilon)\exp\{\frac{x^2}{2\epsilon} J\} = Q_{j,0}(x,\epsilon)^{-1}P_j(x,\epsilon)^{-1}P_\ell(x,\epsilon)Q_{\ell,0}(x,\epsilon) .$$

Hence, the matrix

(6.20)
$$\exp\{-\frac{x^2}{2\epsilon} J\}L(\epsilon)\exp\{\frac{x^2}{2\epsilon} J\} - 1_2$$

is asymptotically zero as $\epsilon \to 0$ in $S_\ell \cap S_j$ uniformly in the domain (6.14-0), where 1_2 is the two-by-two identity matrix.

In the same way (manipulating with the connection formulas (6.11-1) and (6.11-j)), we can prove that the matrix

(6.21)
$$\exp\{\frac{x^2}{2\epsilon} J\}L_1(\epsilon)\exp\{-\frac{x^2}{2\epsilon} J\} - 1_2$$

is asymptotically zero as $\epsilon \to 0$ in $S_\ell \cap S_j$ uniformly in the domain (6.14-1), where

(6.22)
$$L_1(\epsilon) = C(p + \delta_j(\epsilon))L(\epsilon)C(p + \delta_\ell(\epsilon))^{-1} .$$

Also, the matrix

(6.23)
$$\exp\{\frac{x^2}{2\epsilon} J\}L_2(\epsilon)\exp\{-\frac{x^2}{2\epsilon} J\} - 1_2$$

is asymptotically zero as $\epsilon \to 0$ in $S_\ell \cap S_j$ uniformly in the domain (6.14-(-1)), where

(6.24)
$$L_2(\epsilon) = C(-p - \delta_j(\epsilon))^{-1}L(\epsilon)C(-p - \delta_\ell(\epsilon)) .$$

Set

$$(6.25) \quad \begin{cases} L(\varepsilon) = \begin{bmatrix} c_{11}(\varepsilon) & c_{12}(\varepsilon) \\ c_{21}(\varepsilon) & c_{22}(\varepsilon) \end{bmatrix}, \\[3em] L_1(\varepsilon) = \begin{bmatrix} \hat{c}_{11}(\varepsilon) & \hat{c}_{12}(\varepsilon) \\ \hat{c}_{21}(\varepsilon) & \hat{c}_{22}(\varepsilon) \end{bmatrix}, \\[3em] L_2(\varepsilon) = \begin{bmatrix} \tilde{c}_{11}(\varepsilon) & \tilde{c}_{12}(\varepsilon) \\ \tilde{c}_{21}(\varepsilon) & \tilde{c}_{22}(\varepsilon) \end{bmatrix}. \end{cases}$$

Then,

$$(6.26) \quad \hat{c}_{12}(\varepsilon) = \{\lambda_1(p + \delta_j(\varepsilon))c_{11}(\varepsilon) + c_{21}(\varepsilon)\}/\lambda_2(p + \delta_\ell(\varepsilon))$$

$$- \lambda_1(p + \delta_\ell(\varepsilon))\{\lambda_1(p + \delta_j(\varepsilon))c_{12}(\varepsilon) + c_{22}(\varepsilon)\}/\lambda_2(p + \delta_\ell(\varepsilon)) ;$$

and

$$(6.27) \quad \tilde{c}_{21}(\varepsilon) = \lambda_1(-p - \delta_\ell(\varepsilon))c_{11}(\varepsilon) + \lambda_2(-p - \delta_\ell(\varepsilon))c_{12}(\varepsilon)$$

$$- \frac{\lambda_1(-p - \delta_j(\varepsilon))}{\lambda_2(-p - \delta_j(\varepsilon))} \{\lambda_1(-p - \delta_\ell(\varepsilon))c_{21}(\varepsilon) + \lambda_2(-p - \delta_\ell(\varepsilon))c_{22}(\varepsilon)\} .$$

Utilizing the fact that, for any $\varepsilon \in S_\ell \cap S_j$, there exists an $x \in \mathcal{D}_2$ such that

(i) (x, ε) is in the domain (6.14-h),

(ii) $x/\varepsilon^{\frac{1}{2}}$ takes either a real value or a purely imaginary value,

we derive from (6.20), (6.21) and (6.23) that

(1) $c_{11}(\varepsilon) - 1$ and $c_{22}(\varepsilon) - 1$ are asymptotically zero

as $\varepsilon \to 0$ in $S_\ell \cap S_j$;

(2) $|c_{12}(\varepsilon)| \leqq c \exp(-r^2/|\varepsilon|)$, $|c_{21}(\varepsilon)| \leqq c \exp(-r^2/|\varepsilon|)$

for $\varepsilon \in S_\ell \cap S_j$, where c is a positive constant;

(3) $|\hat{c}_{12}(\varepsilon)| \leqq c \exp(-r^2/|\varepsilon|)$ for $\varepsilon \in S_\ell \cap S_j$;

(4) $|\tilde{c}_{21}(\varepsilon)| \leqq c \exp(-r^2/|\varepsilon|)$ for $\varepsilon \in S_\ell \cap S_j$.

Set $\mu(a) = \lambda_1(a)/\lambda_2(s)$. Then,

$$\left| \mu(-p - \delta_\ell(\epsilon))c_{11}(\epsilon) - \mu(-p - \delta_j(\epsilon))c_{22}(\epsilon) \right|$$

$$= \left| \mu(-p - \delta_\ell(\epsilon))\{c_{11}(\epsilon) - c_{22}(\epsilon)\} + \{\mu(-p - \delta_\ell(\epsilon)) - \mu(-p - \delta_j(\epsilon))\}c_{22}(\epsilon) \right|$$

$$\leq \hat{c} \exp(-r^2/|\epsilon|) \quad \text{in} \quad S_\ell \cap S_j$$

for some $\hat{c} > 0$. Since $\mu(-p) \neq 0$, we have

$$(6.28) \qquad |c_{11}(\epsilon) - c_{22}(\epsilon)| \leq c_1 |\delta_\ell(\epsilon) - \delta_j(\epsilon)| + c_2 \exp(-r^2/|\epsilon|)$$

in $S_\ell \cap S_j$ for some $c_1 > 0$ and $c_2 > 0$. On the other hand,

$$\left| \lambda_1(p + \delta_j(\epsilon))c_{11}(\epsilon) - \lambda_1(p + \delta_\ell(\epsilon))c_{22}(\epsilon) \right|$$

$$= \left| \{\lambda_1(p + \delta_j(\epsilon)) - \lambda_1(p + \delta_\ell(\epsilon))\}c_{11}(\epsilon) + \lambda_1(p + \delta_\ell(\epsilon))\{c_{11}(\epsilon) - c_{22}(\epsilon)\} \right|$$

$$\leq \hat{c} \exp(-r^2/|\epsilon|) \quad \text{in} \quad S_\ell \cap S_j$$

for some $\hat{c} > 0$. Since $\lambda_1(p) = 0$ and $\dfrac{d\lambda_1}{da}(p) \neq 0$, we have

$$(6.29) \qquad |\delta_\ell(\epsilon) - \delta_j(\epsilon)| \leq c_3 |\lambda_1(p + \delta_\ell(\epsilon))| |c_{11}(\epsilon) - c_{22}(\epsilon)|$$

$$+ c_4 \exp(-r^2/|\epsilon|) \quad \text{in} \quad S_\ell \cap S_j$$

for some $c_3 > 0$ and $c_4 > 0$. An estimate (6.2) follows from (6.28) and (6.29).

7. <u>Resonance</u>: In this section, we shall prove Theorem 1.2. To do this, we return to Section 5. We proved there that the transformation (5.16) takes the system (5.9) to (5.17) in the domain (5.15). The function $\delta(\epsilon)$ satisfies the condition (5.14). We replace (5.14) by

$$(7.1) \qquad |\delta(\epsilon)| \leq H \exp(-r^2/|\epsilon|) \quad \text{in } S$$

for some positive number H.

Set

$$(7.2) \qquad \begin{cases} \Phi_h(x,\epsilon) = \Lambda(\epsilon) \Psi_h(x/\epsilon^{\frac{1}{2}}, p + \delta(\epsilon)) \ , \\ \tilde{\Phi}_h(x,\epsilon) = \Lambda(\epsilon) \Psi_h(x/\epsilon^{\frac{1}{2}}, p) \ , \qquad (h = -1,0,1,2) \ . \end{cases}$$

Then, $\Phi_h(x,\epsilon)$ (resp. $\tilde{\Phi}_h(x,\epsilon)$) are fundamental matrix solutions of (5.17) (resp. (5.10)) such that

$$(7.3) \qquad \begin{cases} \Phi_0(x,\epsilon) = \Phi_1(x,\epsilon) \mathcal{C}(p + \delta(\epsilon)) \ , \\ \Phi_1(x,\epsilon) = \Phi_2(x,\epsilon) \mathcal{C}(-p - \delta(\epsilon)) \ , \\ \Phi_2(x,\epsilon) = \Phi_{-1}(x,\epsilon) \mathcal{C}(p + \delta(\epsilon)) \ , \\ \Phi_{-1}(x,\epsilon) = \Phi_0(x,\epsilon) \mathcal{C}(-p - \delta(\epsilon)) \ , \end{cases}$$

and

$$(7.4) \qquad \begin{cases} \tilde{\Phi}_0(x,\epsilon) = \tilde{\Phi}_1(x,\epsilon) \mathcal{C}(p) \ , \\ \tilde{\Phi}_1(x,\epsilon) = \tilde{\Phi}_2(x,\epsilon) \mathcal{C}(-p) \ , \\ \tilde{\Phi}_2(x,\epsilon) = \tilde{\Phi}_{-1}(x,\epsilon) \mathcal{C}(p) \ , \\ \tilde{\Phi}_{-1}(x,\epsilon) = \tilde{\Phi}_0(x,\epsilon) \mathcal{C}(-p) \ . \end{cases}$$

Set

$$(7.5) \qquad S(x,\epsilon) = \Phi_0(x,\epsilon) \tilde{\Phi}_0(x,\epsilon)^{-1} \ .$$

Then, the transformation

$$(7.6) \qquad V = S(x,\epsilon) U$$

takes (5.17) to (5.10). Hence, the main part of the proof is to show that $S(x,\epsilon) - 1_2$ is asymptotically zero as $\epsilon \to 0$ in S uniformly in $\mathcal{D}_1$. Note

that $r_1 \sim r$. To do this we manipulate in a way similar to the argument in section 4, utilizing the fact that

(i) $x(\epsilon) \exp\{z^2/\epsilon\}$ and $x(\epsilon) \exp\{-x^2/\epsilon\}$ are asymptotically zero as $\epsilon \to 0$ in S uniformly in $\mathcal{D}_1$;

(ii) $C(p + x(\epsilon))C(p)^{-1} - 1_2 = O(\delta(\epsilon))$;

(iii) $C(-p - x(\epsilon))C(-p)^{-1} - 1_2 = O(\delta(\epsilon))$.

The details are left to the reader.

8. <u>Proof of Theorem 1.1</u>: We shall prove Theorem 1.1 in the case when $\nu = 3$. The general case can be treated in the same manner. We shall consider three sectors S_1, S_2, S_3 as shown in Figure 1.
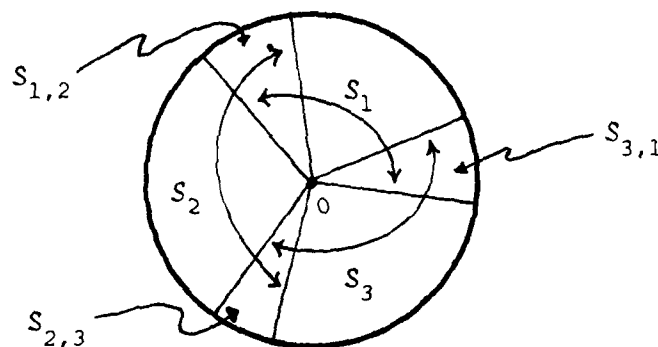

Figure 1

We denote by $S_{1,2}$, $S_{2,3}$, $S_{3,1}$ the intersections $S_1 \cap S_2$, $S_2 \cap S_3$, $S_3 \cap S_1$, respectively.

The three functions $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$, $\delta_3(\varepsilon)$ are holomorphic in S_1, S_2, S_3, respectively. Furthermore,

(8.1) $\delta_j(\varepsilon)$ is asymptotically zero as $\varepsilon \to 0$ in S_j ,

and

(8.2) $\left| \delta_{j+1}(\varepsilon) - \delta_j(\varepsilon) \right| \leqq c_0 \exp(- c_1 / |\varepsilon|^\lambda)$ in $S_{j,j+1}$,

where c_0, c_1, λ are positive numbers, and $S_{3,4} = S_{3,1}$, $\delta_4 = \delta_1$. We shall denote $\delta_{j+1}(\varepsilon) - \delta_j(\varepsilon)$ by $\sigma_j(\varepsilon)$.

We consider a sufficiently small disk:

(8.3) $\mathcal{D} = \{\varepsilon;\ |\varepsilon| \leqq \rho_0\}$.

We choose three line-segments ℓ_1, ℓ_2, ℓ_3 starting from $\varepsilon = 0$ in such a way that

(8.4) $\ell_j \subset S_{j,j+1}$.   (Cf. Figure 2.)

-21-
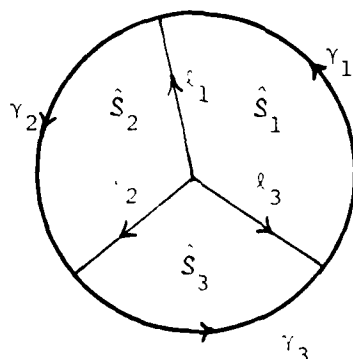
Figure 2

Three line-segments ℓ_1, ℓ_2, ℓ_3 divide the disk $\mathcal{D}$ (cf. (8.3)) into three open sectors $\hat{S}_1$, $\hat{S}_2$, $\hat{S}_3$ (cf. Figure 2). The boundaries of $\hat{S}_1$, $\hat{S}_2$, $\hat{S}_3$ are respectively

$$(8.5\text{-}j) \qquad\qquad \ell_{j-1} + \gamma_j - \ell_j, \qquad j = 1,2,3 ,$$

where $\ell_0 = \ell_3$, and the γ's are circular arcs such that

$$(8.6) \qquad\qquad \gamma_1 + \gamma_2 + \gamma_3 = C = \{\varepsilon; \ |\varepsilon| = \rho_0\} .$$

The line-segments ℓ_j and the circular arcs γ_j are oriented as indicated in Figure 2. We assume that ρ_0 is so small that

$$(8.7) \qquad\qquad \bar{\hat{S}}_j \subset S_j ,$$

where $\bar{\hat{S}}_j$ denotes the closure of $\hat{S}_j$.

Set, for $\varepsilon \in \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3$,

$$(8.8) \qquad\qquad \delta(\varepsilon) = \delta_j(\varepsilon) \quad \text{if} \quad \varepsilon \in \hat{S}_j .$$

Since

$$\frac{1}{2\pi i} \int_{\ell_{j-1}+\gamma_j-\ell_j} \frac{\delta_j(\xi)}{\xi - \varepsilon} \, d\xi = \begin{cases} \delta_j(\varepsilon) & \varepsilon \in \hat{S}_j , \\ 0 & \varepsilon \notin \hat{S}_j , \end{cases}$$

we have

-22-

$$\delta(\epsilon) = \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{\ell_{j-1}+\gamma_j-\ell_j} \frac{\delta_j(\epsilon)}{\xi - \epsilon} \, d\xi \quad \text{in} \quad \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3 .$$

Utilizing

$$\frac{1}{\xi - \epsilon} = \sum_{m=0}^{N} \xi^{-(m+1)} \epsilon^m + \frac{\epsilon^{N+1}}{\xi^{N+1}(\xi - \epsilon)} ,$$

we derive

$$\delta(\epsilon) = \frac{1}{2\pi i} \sum_{m=0}^{N} \left\{ \sum_{j=1}^{3} \int_{\ell_{j-1}+\gamma_j-\ell_j} \xi^{-(m+1)} \delta_j(\xi) d\xi \right\} \epsilon^m$$

$$+ \left\{ \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{\ell_{j-1}+\gamma_j-\ell_j} \frac{\delta_j(\xi)}{\xi^{N+1}(\xi - \epsilon)} \, d\xi \right\} \epsilon^{N+1} .$$

Since $\delta(\epsilon)$ is asymptotically zero as $\epsilon \to 0$ in $\hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3$, the first term must be zero, and hence

$$\delta(\epsilon) = \left\{ \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{\ell_{j-1}+\gamma_j-\ell_j} \frac{\delta_j(\epsilon)}{\xi^{N+1}(\xi - \epsilon)} \, d\xi \right\} \epsilon^{N+1} .$$

Thus we arrive at the following formula:

$$(8.9) \qquad \delta(\epsilon) = \frac{1}{2\pi i} \left\{ \sum_{j=1}^{3} \int_{\ell_j} \frac{\sigma_j(\xi)}{\xi^N(\xi - \epsilon)} \, d\xi + \int_C \frac{\delta(\xi)}{\xi^N(\xi - \epsilon)} \, d\xi \right\} \epsilon^N$$

for $\epsilon \in \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3$ and $N = 1, 2, 3, \ldots$, where $\sigma_j = \delta_{j+1} - \delta_j$.

Construct three open sectors $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3$ as shown in Figure 3, where $0 < \rho_1 < \rho_0$ and θ is a small positive number. Then,

$$\left| \int_C \frac{\delta(\xi)}{\xi^N(\xi - \epsilon)} \, d\xi \right| \leq \frac{C_0}{\rho_0^{N-1}} \frac{1}{\rho_0 - \rho_1}$$
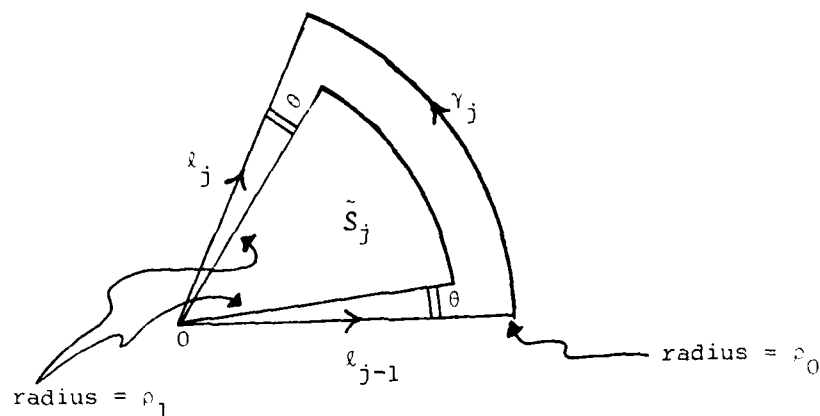
and

Figure 3

$$\left| \int_{\ell_j} \frac{\sigma_j(\xi)}{\xi^N(\xi - \epsilon)} \, d\epsilon \right| \leq \frac{c_0}{\sin \theta} \int_0^{\rho_0} t^{-N-1} \exp(-c_1 t^{-\lambda}) dt$$

$$< \frac{c_0}{\lambda \sin \theta} \int_0^{+\infty} \tau^{(N\lambda^{-1}-1)} \exp(-c_1 \tau) d\tau$$

$$= \frac{c_0}{\lambda \sin \theta} c_1^{-(N/\lambda)} \Gamma(N/\lambda)$$

for $\epsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3$, where c_0 is a positive number. Since

$$\Gamma(N/\lambda) \leq C_1 (N/\lambda)^{(N/\lambda)} e^{-(N/\lambda)}$$

for some $C_1 > 0$, we have

$$(8.10) \qquad \qquad |\delta(\epsilon)| \leq C_2 \left(\frac{|\epsilon|^{\lambda} N}{c_1 \lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)}$$

for $\epsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3$; C_1 is a positive number. For a given ϵ, choose N so that

$$\frac{N}{\lambda} < \frac{c_1}{|\epsilon|^{\lambda}} \leq \frac{N+1}{\lambda} .$$

Then, it follows from (8.10) that

$$(8.11) \qquad \qquad |\delta(\epsilon)| \leq C_2 e^{\frac{1}{\lambda}} \exp(-c_1/|\epsilon|^{\lambda}) .$$

Choosing $\epsilon_1, \epsilon_2, \epsilon_3$ in various ways, we can complete the proof of Theorem 1.1.

-24-

REFERENCES

1) P. P. N. de Groen, The nature of resonance in a singular perturbation problem of turning point type, SIAM J. Math. Anal. 11 (1980), 1-22.

2) N. Kopell, A geometric approach to boundary layer problems exhibiting resonance, SIAM J. Appl. Math. 37 (1979), 436-458.

3) R. McKelvey and R. Bohac, Ackerberg-O'Malley resonance revisited, Rocky Mountain J. Math. 6 (1976), 637-650.

4) B. J. Matkowsky, On boundary layer problems exhibiting resonance, SIAM Review 17 (1975), 82-100.

5) J. P. Ramis, Dévissage Gevrey, Astérisque 59-60 (1978), 173-204.

6) Y. Sibuya, Uniform simplification in a full neighborhood of a transition point, Memoirs, AMS, No. 149, 1974.

7) Y. Sibuya, Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient, North-Holland, 1975.

YS/scr

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER MRC-TSR-2103	2. GOVT ACCESSION NO. AD-A093564	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A THEOREM CONCERNING UNIFORM SIMPLIFICATION AT A TRANSITION POINT AND THE PROBLEM OF RESONANCE -		5. TYPE OF REPORT & PERIOD COVERED Summary Report, no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Yasutaka Sibuya		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 NSF-MCS79-01998
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 (Applied Analysis)
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		12. REPORT DATE August 1980
		13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Analytic continuation
Asymptotic representations in the complex domain
Parabolic cylinder functions
Singular perturbations
Turning point theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
    Given sectors $S_j = \{\varepsilon;\ a_j < \arg \varepsilon < b_j,\ 0 < |\varepsilon| < \rho\}$ $(1 \leq j \leq \nu)$ and

functions δ_j $(1 \leq j \leq \nu)$ such that (i) $\bigcup_j S_j = \{\varepsilon;\ 0 < |\varepsilon| < \rho\}$, (ii) δ_j

is holomorphic in S_j, (iii) δ_j is asymptotically zero as $\varepsilon \to 0$ in S_j,

(iv) $|\delta_j(\varepsilon) - \delta_k(\varepsilon)| \leq c_0 \exp\{-c_1/|\varepsilon|^\lambda\}$ in $S_j \cap S_k$ for some positive numbers

20. ABSTRACT - Cont'd.

c_0, c_1 and λ whenever $S_j \cap S_k \neq \phi$, we prove that $|\delta_j(\varepsilon)| \leq c_2 \exp\{-c_1 / |\varepsilon|^\lambda\}$ in S_j for some positive number c_2. Then, utilizing this result, we prove that Matkowsky-condition implies the resonance in the sense of N. Kopell under a reasonable assumption. The sufficiency of Matkowsky-condition with regard to the Ackerberg-O'Malley resonance has been an open question. This work gives an affirmative answer to this question in a reasonably general case.